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Lesson Plan 3D-printing: binomial formula

topic: proofing the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

curriculum reference:

Die Schülerinnen und Schüler können Terme umformen, auch unter Anwendung der Potenzdarstellung mit positiven ganzzahligen Exponenten.

- Kennen und Anwenden der Potenzdarstellung mit positiven ganzzahligen Exponenten
- Kennen und Anwenden der elementaren Rechenregeln für Potenzen; allenfalls Begründen dieser Regeln
- Umformen von Termen zB durch Ausmultiplizieren, Zusammenfassen, Herausheben, Kürzen (zB $4 \cdot (3x-1) - 6 \cdot (x+4)$; $5 - \frac{x-2}{}$[illegible]; $\left(3x - \frac{1}{}y\right) - \left(\frac{1}{}x + \frac{1}{}y\right)$[illegible]; $4x :$ [illegible]; [illegible])
- Herleiten, grafisches Veranschaulichen und Anwenden der drei binomischen Formeln; allenfalls Herleiten weiterer Rechenregeln (zB $(a+b)^3 = \ldots$)

In the 3rd year of secondary school 1, students are supposed to derive and graphically illustrate the three binomial formulas and in addition to derive further calculation rules like $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

prior knowledge:

Students are familiar with the binomial formulas $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

They can prove these formulas using geometric models and algebra.

learning objectives:

1. *content objectives*

Students will derive the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ by breaking down a cube into geometric components.

They will connect geometric representations with algebraic terms.

They will understand and apply the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

2. *methodological objectives*

Students will work with physical 3D-printed models and a GeoGebra applet to visualize and manipulate the components of the formula.

3. *social objectives*

Students will collaborate in small groups to explore and discuss their observations.

learning outcomes:

- The students can understand how the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ comes about.
- The students can replicate the formula by using geometric figures.

materials needed:

Figure 1: the expanded cube

- 3D-printed cube representing $(a+b)^3$
- 3D-printed components of the expanded cube.
  - 1x a^3
  - 3x a^2b
  - 3x ab^2
  - 1x b^3

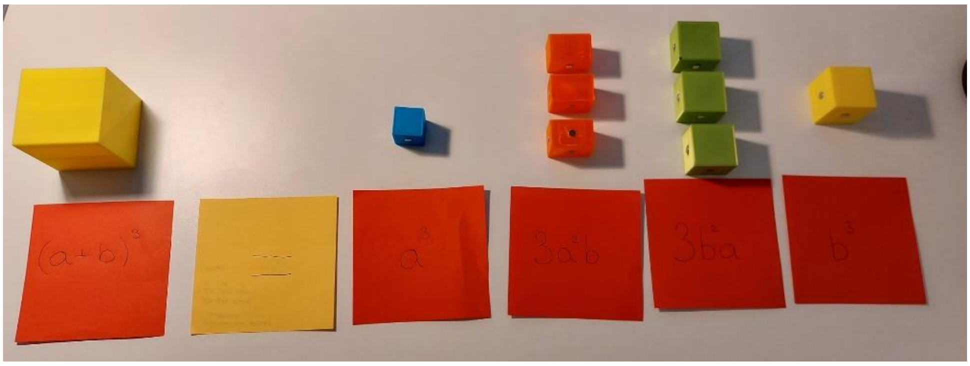

Figure 2: showing that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

- GeoGebra applet: [Binomische Formel "hoch drei" – GeoGebra]

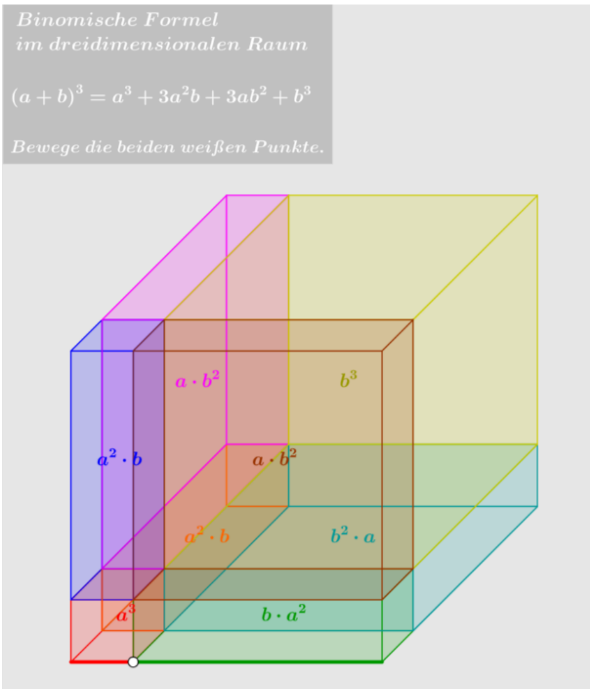

Figure 3: GeoGebra applet

lesson description

time in min	phase	content	comment	materials
10	***introduction***	Recap prior knowledge and introduce the new concept.	-Briefly review the square models for $(a+b)^2$ and the proof (geometric and algebraic). -Pose the question: "What happens when we expand $(a+b)^3$? How can we visualize it?" -Show the 3D-printed cube representing $(a+b)^3$ and explain that today's goal is to break it down and understand its algebraic expansion. -Let the student solve $(a+b)^3$ using their current knowledge	- 3D printed cube $(a+b)^3$
15	***elaboration***	How do the dimensions of each piece relate to the terms in the expansion? Why are there three identical pieces for a^2b and ab^2	-Students work in pairs or small groups (depends on class size) -They disassemble the cube into its components and label each piece with a and b -Groups discuss and try to match the components with the terms of the formula $a^3 + 3a^2b + 3ab^2 + b^3$ -Students compare the cube formed from singular pieces with the cube that is one piece	- 3D printed cube - 3D printed pieces of the eight smaller components of the cube: ○ 1x a^3 ○ 3x a^2b ○ 3x ab^2 ○ 1x b^3

10			-Demonstrate the GeoGebra applet on a projector or individual devices. -Students use the applet to digitally assemble the components into a cube. -Highlight how the algebraic terms arise from the geometric assembly.	-GeoGebra Applet -Laptops/Tablets
10		Connect the exploration to the algebraic representation	-Write $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ on the board and expand step by step, connecting terms to the 3D pieces. -Students identify where each term in the formula comes from and link it to the physical and digital models.	-board -3D printed models -GeoGebra Applet
5	***securing results***	Practice and Reflection	Students solve a few problems applying $(a+b)^3$, e.g., expand and simplify expressions like $(3+x)^3$	-math books

student tasks:

disassembling the 3D Cube

- Students break down the 3D-printed cube $(a+b)^3$, into its eight individual components.
- They measure and label each piece according to the sides a and b
- They match each component to one term in the expanded formula $a^3 + 3a^2b + 3ab^2 + b^3$

group discussion

- In small groups, students discuss the relationships between the physical pieces and the algebraic terms.
- They explore questions like: "Why are there three identical pieces for $3a^2b/3ab^2$?" and "How does the cube's total volume relate to $(a+b)^3$?"

digital assembly with GeoGebra

- Using a GeoGebra applet, students virtually manipulate the pieces to reconstruct the cube.
- They compare the physical and digital assembly processes and reflect on how each helps them understand the formula.

linking algebra and geometry

- Students expand $(a+b)^3$ algebraically on their own and verify that the expanded terms correspond to the components of the cube.

problem solving

- Students apply their understanding to expand expressions such as $(3+x)^3$or solve problems requiring them to identify the volume of specific cube sections.

justification:

3D-Printed Cube and Components

Justification**:** Concrete manipulatives give students a tactile way to explore the formula. Breaking the cube into parts helps them connect the algebraic terms with geometric representations, deepening conceptual understanding.

Why effective**:** Hands-on models cater to kinaesthetic learners and make abstract concepts more accessible.

Group Discussion

Justification: Collaborative exploration encourages students to verbalize and refine their understanding, benefiting from peer insights.

Why effective: Discussion supports the social construction of knowledge and allows students to address misunderstandings collectively.

GeoGebra Applet

Justification: Digital manipulation complements physical models by offering flexibility and precision, reinforcing spatial reasoning.

Why effective: Technology engages visual learners and allows students to experiment without physical constraints, encouraging deeper exploration.

Linking Algebra and Geometry

Justification: Explicitly connecting algebraic terms to geometric models ensures students see how the formula is derived, not just memorized.

Why effective: This approach develops procedural and conceptual knowledge simultaneously.

Problem Solving

Justification: Independent practice ensures students can apply the formula in various contexts, solidifying their learning.

Why effective: Practice moves students from exploration to mastery, building confidence and competence.

hypothetical learning trajectory:

Introduction

Teacher Question: "What does $(a+b)^2$represent geometrically?"

- **Expected Student Answer:** "It's a square with side length $a+\ b$ and the area is split into $a^2, 2ab, b^2$
- **Misconception:** Students may think the cube's volume is only split into three terms.
- **Adaptation:** Use guiding questions like, "What happens when we go from 2D to 3D?"

Disassembling the Cube

Student Observation: "There are eight pieces in total."

- **Expected Reasoning:** Students should recognize that there are 4 types of pieces, and that the number of pieces corresponds to coefficients.
- **Misconception:** Students may mislabel parts or struggle to see why some pieces are counted three times.
- **Adaptation:** Encourage them to compare physical dimensions to algebraic terms, asking, "How many pieces have one side of length a and two sides of length b?"

GeoGebra Applet

Student Interaction: "It's easier to see how the parts fit together digitally."

- **Expected Behaviour:** Students will manipulate the components and note how the volume of each matches its algebraic term.
- **Misconception:** Some students might think the digital model is an approximation rather than a perfect representation.
- **Adaptation:** Emphasize that the applet directly reflects the geometric properties of the physical model.

Linking Algebra and Geometry

Student Question: "Why is there a coefficient of 3 for $3a^2b, 3ab^2$?"

- **Expected Reasoning:** Students recognize that three identical rectangular prisms are formed when $a + b$ is cubed.
- **Misconception:** Students might struggle to see how $3a^2b, 3ab^2$relate to the binomial expansion process.
- **Adaptation:** Revisit the geometric assembly process and connect it step-by-step to the algebraic expansion.

Problem Solving

Expected Performance: Most students will correctly expand expressions like $(3 + x)^3$.

- **Potential Challenge:** Some might forget to multiply coefficients properly or mix up terms or use the two-dimensional binomial formula.
- **Adaptation:** Provide targeted feedback, and if errors persist, use step by step solutions to reinforce patterns.