Enhancing Geometry Learning with GeoGebra: A Study

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Abstract: This study investigates the integration of GeoGebra software and various digital technologies, including augmented reality (AR), virtual reality (VR), and 3D printing, into geometry courses for future mathematics teachers at the Faculty of Education, University of Ostrava. Conducted over two years using a design-based research (DBR) approach, the study involved first- and second-year students training to become primary school mathematics teachers. The primary objectives were to enhance conceptual understanding, improve spatial reasoning, develop algorithmic and critical thinking, and reduce reliance on formal knowledge through a combination of traditional and digital methods. The teaching model was iteratively developed and refined based on continuous feedback and evaluation. Initial activities combined traditional "pencil and paper" techniques with GeoGebra's dynamic capabilities. Advanced features, including GeoGebra applets, AR, VR, and 3D printing, were progressively incorporated to provide more immersive and interactive learning experiences. Data were collected via questionnaires, classroom observations, and reflective journals, offering comprehensive insights into student engagement and learning outcomes. Findings indicate that integrating GeoGebra and digital technologies significantly enhances student engagement, motivation, and understanding of geometric concepts. Students demonstrated improved spatial reasoning and algorithmic thinking, and future teachers gained valuable skills for creating interactive and effective learning environments. Challenges included initial resistance to digital tools and the need for continuous adaptation based on feedback. The study concludes that incorporating GeoGebra and digital technologies into geometry education can transform traditional teaching methods, making lessons more interactive, engaging, and effective. Future research should focus on long-term retention of geometric concepts, larger and more diverse sample groups, and further integration of advanced digital features and cross-disciplinary applications. This innovative approach holds significant potential for enhancing geometry education and preparing future educators with practical, modern teaching strategies.

Keywords: Augmented Reality in Education, GeoGebra, Teacher Training, Geometry Courses, Algorithmic Thinking.

1. Introduction

Geometry is fundamental in mathematics education, crucial for developing spatial reasoning, problem-solving, and logical thinking. Traditional methods of teaching geometry often rely heavily on "pencil and paper" techniques. While foundational, these methods can limit students' engagement and understanding by focusing on rote learning and procedural tasks. This approach may not effectively convey the dynamic and interconnected nature of geometric concepts, leading to difficulties in visualizing and comprehending abstract relationships. Consequently, students may struggle to apply their knowledge in practical contexts.

To address these challenges, integrating innovative tools like GeoGebra can transform the teaching and learning of geometry. GeoGebra, a dynamic mathematics software, offers a multifaceted approach by allowing students to explore and manipulate geometric figures interactively. This provides immediate visual feedback, fosters a deeper understanding of geometric relationships, and enhances algorithmic thinking and problem-solving skills. Additionally, advanced features such as augmented reality (AR), virtual reality (VR), and 3D printing can provide more immersive and interactive learning experiences.

The primary aim of this study is to develop and refine a teaching model for geometry that integrates traditional methods with the dynamic capabilities of GeoGebra and other digital technologies. This model seeks to reduce students' reliance on formal knowledge, enhance their conceptual understanding of geometry, improve spatial reasoning and algorithmic thinking, and foster critical thinking through dynamic exploration and manipulation. Additionally, it aims to prepare future mathematics teachers with practical skills for creating engaging learning environments.

2. Research Objectives

The primary aim of this study is to develop and refine a teaching model for geometry that integrates traditional "pencil and paper" methods with the dynamic capabilities of GeoGebra and various digital technologies, including virtual reality (VR), augmented reality (AR), applets, and 3D printing. This innovative approach seeks to enhance the overall learning experience and equip future mathematics teachers with the skills and tools necessary for effective teaching. The specific research objectives are as follows:

1. *Reduce Reliance on Formal Knowledge*: Traditional geometry education often emphasizes rote memorization and procedural tasks. This study aims to reduce students' reliance on formal knowledge by incorporating GeoGebra's interactive features, which encourage conceptual understanding through exploration and manipulation of geometric figures.
2. *Enhance Conceptual Understanding of Geometry*: By using GeoGebra applets, augmented reality, and dynamic visualization tools, the study seeks to improve students' grasp of geometric concepts. These tools help bridge the gap between abstract theories and practical applications, making it easier for students to visualize and understand complex geometric relationships.
3. *Improve Spatial Reasoning and Algorithmic Thinking*: The integration of GeoGebra and VR, AR, and 3D printing technologies aims to develop students' spatial reasoning skills by allowing them to interact with 3D shapes and perform dynamic constructions. Additionally, these technologies promote algorithmic thinking by enabling students to experiment with different geometric scenarios and solve problems through step-by-step exploration.
4. *Foster Critical Thinking through Dynamic Exploration*: GeoGebra's dynamic environment allows students to engage in inquiry-based learning, where they can pose questions, test hypotheses, and explore geometric properties. This approach encourages critical thinking and helps students develop a deeper understanding of the subject matter.
5. *Prepare Future Mathematics Teachers with Practical Skills*: The study aims to equip future mathematics teachers with a comprehensive toolkit of teaching methods and digital tools. By experiencing a variety of teaching techniques and technologies, students will be better prepared to create engaging and effective learning environments in their future classrooms.
6. *Create Engaging Learning Environments*: By combining traditional teaching methods with innovative digital tools such as GeoGebra applets, AR, VR, and 3D printing, the study seeks to make geometry lessons more interactive, engaging, and personalized. This approach aims to increase student motivation and participation, ultimately leading to improved learning outcomes.
7. *Evaluate the Effectiveness of the Teaching Model*: The study will employ a design-based research (DBR) approach to iteratively design, test, and refine the teaching model. Through continuous feedback and evaluation, the study aims to identify best practices and optimize the integration of GeoGebra and related technologies in geometry education.

By addressing these objectives, the study aims to transform traditional geometry teaching methods, enhance student engagement and understanding, and prepare future educators to effectively use modern digital tools in their classrooms.

3. Theoretical Framework

This study is grounded in the principles of Inquiry-Based Learning (IBL), Design-Based Learning (DBL), and social constructivism.

- **Inquiry-Based Learning (IBL)**: Encourages students to take an active role in their learning through questioning, exploration, and problem-solving, promoting deeper understanding through discovery and hands-on activities (Piaget, 1971; Vygotsky, 1978).
- **Design-Based Learning (DBL)**: Integrates the design process into educational activities, fostering creativity, problem-solving, and practical application of knowledge through iterative cycles of designing, testing, and refining solutions (Haas, Kreis, & Lavicza, 2021).
- **Social Constructivism**: Emphasizes learning through social interaction and collaboration, with knowledge constructed through shared experiences and guided interaction (Kostrub, & Ostradicky, 2019; Vygotsky, 1978).

These frameworks inform the integration of GeoGebra, AR, VR, and 3D printing to create an engaging and effective learning environment that enhances understanding of geometry while developing critical thinking, spatial reasoning, and algorithmic skills (Bohdal, 2019; Dana-Picard et al., 2021; Kusumah, Kustiawati, & Herman, 2020).

4. Methodology

The study employed a design-based research (DBR) approach, characterized by iterative cycles of designing, testing, and refining educational interventions. This approach is particularly suitable for educational settings as it allows for continuous improvement of teaching practices based on real-time feedback and observations (Lavicza et al., 2022). The DBR process is guided by principles of social constructivism and inquiry-based learning (IBL), ensuring that the interventions are both theoretically grounded and practically relevant (Piaget, 1971; Vygotsky, 1978).

Participants in this study were students enrolled in the Mathematics Education program at the Faculty of Education, University of Ostrava. Each academic year, a single study group consisting of 19 to 25 students participated in the Synthetic Geometry course. These students, training to become primary school mathematics teachers, provided a relevant and focused sample for evaluating the effectiveness of the teaching model in preparing future educators.

Data were collected using a variety of methods to capture a comprehensive understanding of the teaching model's effectiveness. Questionnaires were used to gather students' feedback on their learning experiences, understanding of geometric concepts, and engagement levels. Both pre- and post-intervention questionnaires were administered to measure changes over time. Observations were conducted during classroom sessions to monitor student interactions, use of GeoGebra, and engagement with the tasks. Teacher maintained reflective journals to document their observations and reflections on the teaching process (Kemmis, McTaggart, & Nixon, 2014). Additionally, both students and teacher were encouraged to keep reflective journals, with students reflecting on their learning experiences and challenges faced, while teacher documented their observations, adjustments made to the teaching model, and reflections on its effectiveness (Švaříček & Šeďová, 2007).

The study employed both qualitative and quantitative data collection methods. Qualitative data included feedback from questionnaires, observations, and reflective journals, providing insights into students' experiences, challenges, and perceptions of the teaching model (Kostrub, & Ostradicky, 2019). These qualitative methods allowed for a deeper understanding of the contextual and experiential aspects of the teaching and learning process. Quantitative data included pre- and post-intervention test scores to measure improvements in students' understanding of geometric concepts and their spatial reasoning abilities. Statistical analyses were conducted to evaluate the significance of the observed changes, providing measurable evidence of the effectiveness of the teaching interventions.

The DBR process was implemented in several phases, each building on the feedback and data collected from the previous phase. The initial implementation phase involved introducing GeoGebra and basic training sessions for students, with initial activities designed to combine traditional "pencil and paper" methods with GeoGebra. The refinement phase included adjustments based on initial feedback, with enhanced activities and tasks using GeoGebra to address identified challenges. Additional training sessions were provided to improve proficiency with GeoGebra. The advanced implementation phase involved more complex tasks and projects integrating GeoGebra, the introduction of gamification elements, and collaborative learning activities. Continuous collection of feedback and observations allowed for further refinements (Lavicza et al., 2022).

The development of the teaching model focused on combining traditional "pencil and paper" methods with GeoGebra, emphasizing foundational skills and dynamic visualizations. Students performed manual constructions and used GeoGebra to explore and verify their results, with the model refined through multiple iterations based on feedback from students and teacher. Adjustments included balancing manual and digital tasks, enhancing scaffolding, and integrating advanced GeoGebra features such as AR and 3D printing. In addition to GeoGebra, VR technologies were integrated into the teaching model, allowing students to explore 3D geometric shapes and solve complex construction tasks, thereby enhancing their spatial reasoning and engagement (Kusumah, Kustiawati, & Herman, 2020).

The specific activities and tasks designed for the teaching model included basic constructions where students performed geometric constructions using both traditional methods and GeoGebra, dynamic explorations where students manipulated geometric figures in GeoGebra to explore properties and relationships dynamically, collaborative projects where students used GeoGebra to model and solve complex geometric problems, and gamification activities incorporating game elements to make learning more engaging.

Through this comprehensive and iterative methodological approach, the study aimed to create an effective and engaging teaching model that leverages the strengths of both traditional and digital methods to enhance geometry education.

5. Development of the Teaching Model

The development of the teaching model was an iterative process, leveraging the principles of design-based research (DBR) to continuously refine and improve the integration of GeoGebra and various digital technologies into the geometry curriculum. This process was guided by feedback from students and teacher, ensuring the model remained responsive to their needs and effectively enhanced learning outcomes.

5.1 Initial Model Design

The initial design of the teaching model combined traditional "pencil and paper" methods with the dynamic capabilities of GeoGebra. Traditional methods focused on foundational skills such as manual constructions, proofs, and the development of geometric intuition. These methods are essential for students to grasp the basic principles and logic underlying geometric concepts. However, they often lack the ability to provide immediate visual feedback and interactive exploration. To address these limitations, GeoGebra was introduced as a digital tool to enhance visualization and interactivity.

The initial activities were designed to transition smoothly from traditional methods to incorporating GeoGebra. For example, students started with basic constructions using a compass and straightedge, then replicated these constructions in GeoGebra. This dual approach reinforced their understanding and allowed them to compare the manual and digital processes. Teaching strategies included guided discovery, where students were encouraged to explore geometric properties and relationships independently using GeoGebra. Teacher provided scaffolding through targeted questions and prompts, helping students make connections between their manual constructions and the dynamic models in GeoGebra (Hohenwarter & Jones, 2007).

5.2 Iterative Improvement Process

The development of the teaching model followed an iterative process typical of DBR. This process involved multiple phases of implementation, feedback, and refinement:

- *Pilot Phase*: The initial model was tested with a first group of students to gather preliminary feedback and identify potential issues. This phase focused on evaluating the effectiveness of combining traditional methods with GeoGebra and observing student engagement and understanding.
- *First Iteration*: Based on feedback from the pilot phase, adjustments were made to the activities and teaching strategies. This included refining the balance between manual and digital tasks and enhancing the scaffolding provided by teacher. The revised model was then implemented with a new group of students in the following academic year.
- *Second Iteration*: Further feedback was collected through observations, student questionnaires, and teacher reflections. This data was used to make additional refinements, such as adjusting the complexity of tasks and improving the integration of GeoGebra features. The model was continuously tested and improved over several iterations (Lavicza et al., 2022).

Feedback played a crucial role in the iterative improvement process. Teacher kept reflective journals, noting observations and insights from each lesson. Student feedback was gathered through questionnaires focusing on their experiences and perceptions of the learning activities. This feedback loop allowed for ongoing adjustments and refinements. For example, if students reported difficulty in transitioning between manual and digital methods, additional support and practice activities were introduced. If teacher observed that certain GeoGebra features were underutilized, targeted training sessions were conducted to enhance their integration (Kostrub, & Ostradicky, 2019).

5.3 Incorporation of Advanced GeoGebra Features

As the teaching model evolved, more advanced features of GeoGebra were incorporated to further enhance student learning. These included:

- *GeoGebra Augmented Reality (AR):* GeoGebra AR allowed students to visualize and interact with 3D geometric shapes in their physical environment. This immersive experience significantly improved students' spatial reasoning and made abstract concepts more tangible (Lavicza et al., 2022).
- *GeoGebra for Constructive Tasks*: Digital construction tasks using GeoGebra reinforced students' manual skills while introducing dynamic methods of verifying and exploring their constructions. This promoted critical thinking and precision in their work (Kusumah, Kustiawati, & Herman, 2020).

- *GeoGebra Applets*: Interactive modules enabled students to manipulate geometric figures and observe transformations in real-time, fostering a deeper understanding of geometric relationships (Hohenwarter & Jones, 2007).
- *3D Printing:* Integrating 3D printing with GeoGebra allowed students to create physical models of geometric shapes, providing a hands-on experience that further reinforced their understanding of spatial relationships and geometric properties (Lavicza et al., 2022).
- *GeoGebra with Virtual Reality (VR*): In this phase, we incorporated GeoGebra applets into virtual reality environments. Using VR software, students were able to immerse themselves in 3D geometric worlds, exploring and manipulating shapes in a more engaging and interactive manner. The VR integration significantly enhanced spatial reasoning and provided a unique, immersive learning experience. Additionally, we employed gamification techniques within the VR environment to make learning more engaging. For instance, students participated in virtual challenges and games that required them to solve geometric problems and construct shapes using GeoGebra in VR. This gamified approach not only increased motivation and participation but also reinforced their understanding of geometric concepts through active involvement (Berger-Haladová, & Ferko, 2019).

5.4 Gamification and Collaborative Learning

An innovative element of the teaching model was the incorporation of gamification and collaborative learning. For example, students were divided into groups, each focusing on different geometric shapes like squares, rhombuses, and ellipses. They described these shapes using their characteristic properties and then challenged other groups to identify them based on these descriptions. This playful, competitive element made learning more engaging and reinforced their understanding of geometric properties through peer interaction (Berger-Haladová, & Ferko, 2019).

5.5 Continuous Refinement and Feedback

The iterative nature of the DBR approach allowed for continuous refinement of the teaching model based on feedback from students and teacher. Each phase of implementation provided valuable insights into the effectiveness of the activities and the overall model. Adjustments were made to improve the balance between traditional and digital methods, enhance the scaffolding provided by teacher, and increase the complexity of tasks as students' skills developed. Feedback from reflective journals and questionnaires played a critical role in shaping these refinements (Kemmis, McTaggart, & Nixon, 2014; Švaříček & Šeďová, 2007).
The final iteration of the teaching model demonstrated significant improvements in both student engagement and understanding of geometric concepts. The combination of "pencil and paper" methods with GeoGebra proved to be highly effective, making lessons more interactive, engaging, and personalized. Students not only acquired a deeper understanding of geometry but also developed essential skills such as problem-solving, critical thinking, and collaboration. The positive outcomes of the refined teaching model highlight the potential of integrating digital tools like GeoGebra into regular geometry teaching to enhance educational experiences and outcomes.

6. Activities and Their Impact

The teaching model developed in this study incorporated a variety of activities designed to integrate traditional methods with the dynamic capabilities of GeoGebra and other digital technologies. These activities were carefully crafted to enhance student engagement, understanding, and retention of geometric concepts. The impact of these activities was assessed through continuous feedback and iterative refinements, ensuring their effectiveness in achieving the study's educational objectives.

6.1 Basic Constructions

One of the foundational activities involved basic geometric constructions using both traditional "pencil and paper" methods and GeoGebra. Students initially performed constructions manually, using tools like compasses and straightedges, to develop their foundational skills and geometric intuition. They then replicated these constructions digitally in GeoGebra, which allowed them to explore and verify their results dynamically. This dual approach reinforced their understanding and highlighted the advantages of digital tools in visualizing geometric relationships.

The impact of these basic construction activities was significant. Students reported that the immediate visual feedback provided by GeoGebra helped them grasp concepts more quickly and accurately. They appreciated the ability to manipulate geometric figures interactively, which made abstract concepts more tangible and easier to understand. This activity also promoted critical thinking and problem-solving skills, as students could test different scenarios and observe the outcomes in real-time (Schmidthaler et al., 2023).

6.2 Dynamic Explorations

Dynamic exploration tasks allowed students to manipulate geometric figures in GeoGebra to explore properties and relationships dynamically. For instance, students could change the dimensions of shapes, transform them, and observe how these changes affected their properties. These tasks encouraged students to engage deeply with geometric concepts, fostering a deeper understanding through active experimentation.
The impact of dynamic explorations was reflected in enhanced spatial reasoning and algorithmic thinking among students. By actively engaging with geometric figures and observing their transformations, students developed a more intuitive understanding of geometric properties and relationships. This hands-on approach also increased their confidence in tackling complex geometric problems and applying their knowledge in new contexts (Hohenwarter & Jones, 2007).

6.3 Collaborative Projects

Collaborative projects were an integral part of the teaching model, promoting teamwork and communication skills. In these projects, students worked in groups to model and solve complex geometric problems using GeoGebra. For example, they might be tasked with designing a geometric structure that meets specific criteria, requiring them to apply various geometric principles and use GeoGebra to visualize and test their designs.
The collaborative projects had a positive impact on student engagement and learning outcomes. Working in groups allowed students to share ideas, learn from each other, and approach problems from different perspectives. This collaborative learning environment fostered a sense of community and support, making the learning experience more enjoyable and effective. Additionally, the projects helped students develop essential skills such as critical thinking, problem-solving, and the ability to communicate mathematical ideas clearly and effectively (Kostrub, & Ostradicky, 2019).

6.4 Gamification Elements

Gamification elements were incorporated into the teaching model to make learning more engaging and interactive. For instance, students participated in competitive challenges where they described geometric shapes using their properties and then challenged other groups to identify them based on these descriptions. This playful, competitive element added excitement to the learning process and motivated students to participate actively.
The impact of gamification was evident in the increased motivation and enthusiasm among students. The competitive and interactive nature of these activities made learning more enjoyable and engaging. Students reported that they looked forward to these challenges and found them helpful in reinforcing their understanding of geometric concepts. Gamification also promoted collaborative learning, as students worked together to solve problems and achieve common goals (Berger-Haladová, & Ferko, 2019).

6.5 Integration of Advanced GeoGebra Features

The teaching model also incorporated advanced features of GeoGebra, such as Augmented Reality (AR), Virtual Reality (VR), and 3D printing, to provide more immersive and hands-on learning experiences. In the AR activities, students used GeoGebra AR to visualize and interact with 3D geometric shapes in their physical environment, enhancing their spatial reasoning skills. In the VR activities, students explored 3D geometric worlds and participated in gamified challenges, further enhancing their engagement and understanding. The integration of 3D printing allowed students to create physical models of geometric shapes, providing a tangible representation of abstract concepts.
The impact of these advanced features was substantial. Students reported that the immersive experiences provided by AR and VR made learning more engaging and effective. The ability to interact with 3D shapes in a virtual environment helped them develop a better understanding of spatial relationships and geometric properties. The use of 3D printing provided a hands-on experience that reinforced their learning and made abstract concepts more concrete (Lavicza et al., 2022).

Overall, the activities designed for the teaching model had a significant positive impact on student engagement, understanding, and retention of geometric concepts. The combination of traditional methods with the dynamic capabilities of GeoGebra and other digital technologies provided a comprehensive approach to teaching geometry, enhancing the learning experience and preparing students for future success in their mathematical education.

7. Findings

The integration of GeoGebra and various digital technologies into the geometry curriculum significantly improved student engagement, understanding, and retention of geometric concepts. The iterative design-based research (DBR) approach allowed for continuous refinement of the teaching model, resulting in substantial educational benefits.

7.1 Enhanced Engagement and Motivation

The dynamic and interactive nature of GeoGebra, along with the incorporation of advanced features like Augmented Reality (AR) and Virtual Reality (VR), significantly increased student engagement and motivation. Students reported higher levels of interest and enthusiasm during lessons, attributing this to the ability to manipulate geometric figures interactively and explore them in immersive environments. The gamification elements further boosted motivation, as students found the competitive and playful aspects of learning to be exciting and stimulating (Berger-Haladová, & Ferko, 2019).

7.2 Improved Understanding of Geometric Concepts

The integration of GeoGebra significantly enhanced students' understanding of geometric concepts. The immediate visual feedback and dynamic manipulation capabilities of GeoGebra allowed students to explore geometric properties and relationships more deeply. Activities involving GeoGebra applets and dynamic explorations helped students visualize transformations, symmetries, and other geometric operations, leading to a more intuitive and comprehensive grasp of these concepts (Hohenwarter & Jones, 2007). The use of AR and VR further solidified this understanding by providing a tangible and immersive experience, making abstract concepts more concrete and easier to comprehend (Bohdal, 2019).

7.3 Development of Spatial Reasoning and Algorithmic Thinking

The study found that the integration of GeoGebra and digital technologies significantly improved students' spatial reasoning and algorithmic thinking skills. Tasks that required students to manipulate 3D shapes in AR and VR environments enhanced their ability to visualize spatial relationships and understand geometric properties from multiple perspectives. Additionally, the iterative design tasks and problem-solving activities promoted algorithmic thinking, as students engaged in step-by-step exploration and verification of geometric constructions (Haas, Kreis, & Lavicza, 2021).

7.4 Positive Impact on Critical Thinking and Problem-Solving Skills

The teaching model promoted critical thinking and problem-solving skills through activities that encouraged inquiry-based learning and collaborative projects. Students were challenged to pose questions, test hypotheses, and explore geometric properties using GeoGebra's dynamic tools. The collaborative projects further enhanced these skills by fostering teamwork and communication, as students worked together to solve complex geometric problems and share their findings. This collaborative learning environment not only improved their understanding of geometry but also developed their ability to think critically and solve problems effectively (Kostrub, & Ostradicky, 2019).

7.5 Preparation for Future Teaching Careers

Future mathematics teachers who participated in the study developed a comprehensive toolkit of teaching methods and digital tools, preparing them for their future roles as educators. By experiencing a variety of teaching techniques and technologies, they gained practical skills in creating engaging and effective learning environments. The iterative refinement of the teaching model ensured that they were well-equipped to integrate modern digital tools, such as GeoGebra, AR, VR, and 3D printing, into their future classrooms. This preparation is crucial for developing innovative teaching practices that can adapt to diverse educational settings and meet the needs of 21st-century learners (Leoste et al., 2022).

7.6 Challenges and Limitations

While the study demonstrated significant positive outcomes, it also identified several challenges and limitations. One of the main challenges was the initial resistance from some students who were unfamiliar with using digital tools in their learning process. This resistance was mitigated through additional training sessions and ongoing support, which helped students become more comfortable and proficient with GeoGebra and other technologies. Another challenge was the need for continuous adaptation of the teaching model based on feedback from students and teacher. This iterative process required considerable time and effort to analyze feedback and implement changes effectively (Kemmis, McTaggart, & Nixon, 2014).

A limitation of the study is the relatively small sample size, as the study group consisted of a single cohort of 19 to 25 students each academic year. Future research should include larger and more diverse sample groups to validate the findings and ensure their broader applicability. Additionally, while the study focused on the short-term impact of the teaching model, further research is needed to investigate the long-term retention of geometric concepts taught using GeoGebra and related technologies.

The findings of this study highlight the transformative potential of integrating GeoGebra and digital technologies into geometry education. The teaching model significantly enhanced student engagement, understanding, and retention of geometric concepts, while also developing essential skills such as spatial reasoning, critical thinking, and problem-solving. The positive outcomes demonstrate that combining traditional methods with innovative digital tools can create a more interactive, engaging, and effective learning environment, ultimately preparing future educators to meet the challenges of modern educational settings.

7.7 Detailed Analysis of Feedback Data

The feedback data collected throughout the study provided valuable insights into students' experiences and perceptions of the teaching model. The key trends and conclusions are as follows:

- *Enhanced Understanding*: Most students reported a high level of understanding of the lectures and course requirements, attributed to the clear structure and interactive elements of GeoGebra.
- *Adaptation to Digital Tools*: Initial challenges in adapting to digital tools were mitigated through additional training sessions and ongoing support, leading to increased comfort and proficiency with GeoGebra.
- *Increased Engagement*: Students consistently demonstrated high levels of engagement and motivation, particularly appreciating the hands-on experience provided by GeoGebra and the immersive features of AR and VR.
- *Improved Retention*: There were indications of better retention of geometric concepts, suggesting the effectiveness of the teaching model in promoting lasting understanding.
- *Positive Reception of Advanced Features*: Students found the use of advanced GeoGebra features, such as AR and 3D printing, highly beneficial for enhancing spatial reasoning and making abstract concepts more tangible.
- *Overall Satisfaction*: Satisfaction levels remained high throughout the study, with students highlighting the combination of interactive digital tools and collaborative learning activities as key factors in their positive experience.

Experience with VR in Geometry Learning

- *Immersive Learning*: VR provided a powerful and immersive way to visualize and understand 3D geometric shapes, significantly enhancing spatial reasoning skills.
- *Gamification and Engagement*: The gamified elements within the VR environment made learning more engaging and enjoyable, motivating students to participate actively and collaboratively.

Overall Implications

- *Enhanced Student Engagement and Motivation*: The integration of GeoGebra and VR technologies significantly boosted student engagement and motivation, making learning more enjoyable and accessible.
- *Improved Understanding and Retention of Geometric Concepts*: Students demonstrated a deeper understanding and better retention of geometric concepts, bridging the gap between theoretical knowledge and practical application.
- *Development of Critical Skills*: The teaching model promoted the development of critical thinking, problem-solving, and spatial reasoning skills.
- *Preparation for Future Teaching Careers*: Future mathematics teachers benefited from the comprehensive toolkit of teaching methods and digital tools, preparing them for innovative teaching practices in diverse educational settings.

These positive outcomes demonstrate that combining traditional methods with innovative digital tools can create a more interactive, engaging, and effective learning environment.

8. Discussion and Conclusion

The integration of GeoGebra and various digital technologies into the geometry curriculum has demonstrated significant benefits in enhancing student engagement, understanding, and retention of geometric concepts. The study's iterative design-based research (DBR) approach allowed for continuous refinement of the teaching model, resulting in a robust and effective educational tool.

The dynamic and interactive nature of GeoGebra, augmented by advanced features such as Augmented Reality (AR) and Virtual Reality (VR), significantly boosted student engagement and motivation. The ability to manipulate geometric figures interactively and explore them in immersive environments made learning more enjoyable and accessible. The inclusion of gamification elements further increased motivation, as students found the competitive and playful aspects of learning to be stimulating and rewarding (Berger-Haladová, & Ferko, 2019).

Students demonstrated a deeper understanding and better retention of geometric concepts through the use of GeoGebra. The immediate visual feedback and dynamic manipulation capabilities allowed for a more intuitive grasp of geometric properties and relationships. Activities involving GeoGebra applets and dynamic explorations helped students visualize transformations and symmetries effectively. The use of AR and VR provided a tangible and immersive experience, making abstract concepts more concrete and easier to comprehend (Hohenwarter & Jones, 2007).

The study found significant improvements in students' spatial reasoning and algorithmic thinking skills. Tasks that required manipulation of 3D shapes in AR and VR environments enhanced their ability to visualize spatial relationships and understand geometric properties from multiple perspectives. Additionally, iterative design tasks and problem-solving activities promoted algorithmic thinking, as students engaged in step-by-step exploration and verification of geometric constructions (Haas, Kreis, & Lavicza, 2021).

The teaching model fostered critical thinking and problem-solving skills through inquiry-based learning and collaborative projects. Students were encouraged to pose questions, test hypotheses, and explore geometric properties using GeoGebra's dynamic tools. Collaborative projects further enhanced these skills by fostering teamwork and communication, allowing students to work together to solve complex geometric problems and share their findings. This collaborative learning environment not only improved their understanding of geometry but also developed their ability to think critically and solve problems effectively (Kostrub, & Ostradicky, 2019).

The study effectively prepared future mathematics teachers by providing them with a comprehensive toolkit of teaching methods and digital tools. By experiencing a variety of teaching techniques and technologies, students gained practical skills in creating engaging and effective learning environments. The iterative refinement of the teaching model ensured that future educators were well-equipped to integrate modern digital tools, such as GeoGebra, AR, VR, and 3D printing, into their classrooms. This preparation is crucial for developing innovative teaching practices that can adapt to diverse educational settings and meet the needs of 21st-century learners (Lavicza et al., 2022).

While the study demonstrated significant positive outcomes, it also identified several challenges and limitations. One of the main challenges was the initial resistance from some students who were unfamiliar with using digital tools in their learning process. This resistance was mitigated through additional training sessions and ongoing support, which helped students become more comfortable and proficient with GeoGebra and other technologies. Another challenge was the need for continuous adaptation of the teaching model based on feedback from students and teacher. This iterative process required considerable time and effort to analyze feedback and implement changes effectively (Kemmis, McTaggart, & Nixon, 2014).

A limitation of the study is the relatively small sample size, as the study group consisted of a single cohort of 19 to 25 students each academic year. Future research should include larger and more diverse sample groups to validate the findings and ensure their broader applicability. Additionally, while the study focused on the short-term impact of the teaching model, further research is needed to investigate the long-term retention of geometric concepts taught using GeoGebra and related technologies.

In conclusion, the findings of this study highlight the transformative potential of integrating GeoGebra and digital technologies into geometry education. The teaching model significantly enhanced student engagement, understanding, and retention of geometric concepts, while also developing essential skills such as spatial reasoning, critical thinking, and problem-solving. These positive outcomes demonstrate that combining traditional methods with innovative digital tools can create a more interactive, engaging, and effective learning environment, ultimately preparing future educators to meet the challenges of modern educational settings.

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